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The cycle structure of regular multipartite tournaments

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Abstract

A multipartite tournament is an orientation of a complete multipartite graph. A tournament is a multipartite tournament, each partite set of which contains exactly one vertex. Alspach (Canad. Math. Bull. 10 (1967) 283) proved that every regular tournament is arc-pancyclic. Although all partite sets of a regular multipartite tournament have the same cardinality, Alspach's theorem is not valid for regular multipartite tournaments. In this paper, we prove that if the cardinality common to all partite sets of a regular n-partite ($n \geqslant 3$) tournament T is odd, then every arc of T is in a cycle that contains vertices from exactly m partite sets for all $m \in \{3, 4, \ldots, n\}$. This result extends Alspach's theorem for regular tournaments to regular multipartite tournaments. We also examine the structure of cycles through arcs in regular multipartite tournaments.

Keywords: Cycle; Regularity; Multipartite tournament

1. Introduction

We consider finite digraphs without loops and multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we say that x *dominates* y. More generally, if A and B are two disjoint subdigraphs of D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B and denote it by $A \rightarrow B$. The *outset* $N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *inset* $N^-(x)$ is the set of vertices dominating x. A digraph D is said to be *regular* if there is an integer r such that $|N^+(x)| = |N^-(x)| = r$ holds for every $x \in V(D)$.

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Let D be a digraph. If we replace every arc xy of D by yx, then we call the resulting digraph, denoted by D^{-1}, the *converse digraph* of D. For a subset A of $V(D)$, the subdigraph induced by A in D is denoted by $D\langle A\rangle$.

Cycles and paths are always simple and directed. A k-cycle is a cycle of length k. A digraph D is *strongly connected* (or just *strong*) if for every ordered pair of distinct vertices (x, y) there exists a path from x to y in D.

A digraph obtained by replacing each edge of a complete n-partite graph with an arc is called a *multipartite tournament*, and a tournament is an n-partite tournament having exactly n vertices. One can easily show that all partite sets of a regular multipartite tournament have the same number of vertices.

Tournaments have a rich structure. For example, Moon [9] proved that if T is a strong tournament on n vertices, then every vertex of T is in an m-cycle for all $m \in \{3, 4, \dots, n\}$, and Alspach [1] proved that every arc of a regular tournament on $n \geqslant 3$ vertices is in an m-cycle for all $m \in \{3, 4, \dots, n\}$, i.e., it is *arc-pancyclic*.

The structure of cycles in multipartite tournaments has been well studied (see [3–8,10]). Examples of Bondy [2] show that Moon's theorem is not valid for strong multipartite tournaments in general. However, from different extensions of the notion of a cycle in a tournament, one obtains various generalizations of Moon's theorem to multipartite tournaments (see [4,6]).

Goddard and Oellermann [4] proved that every vertex of a strongly connected n-partite ($n \geqslant 3$) tournament is in a cycle that contains vertices from exactly m partite sets for all m with $3 \leqslant m \leqslant n$.

However, the result of Alspach above is not valid for regular n-partite tournaments.

For example, let H_0 be the oriented 4-cycle. The regular 6-partite tournament H_1 in Fig. 1 consists of three copies of H_0 with the arcs between them oriented clockwise. Recursively, for $n \geqslant 2$, H_n consists of three copies of H_{n-1} with clockwise oriented arcs between them. The $2 \cdot 3^n$-partite tournament H_n is of order $4 \cdot 3^n$. Clearly, any arc of an induced 4-cycle H_0 in H_n is not in a cycle containing vertices from exactly 3 partite sets, in particular, it is not in a 3-cycle.

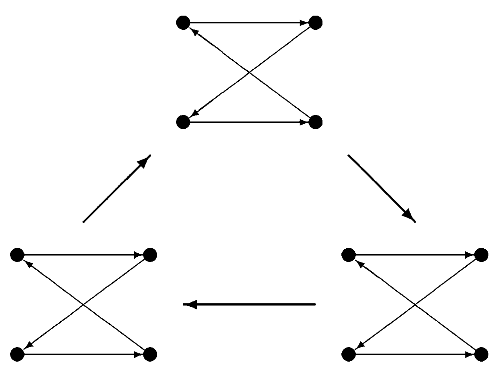

Fig. 1. The regular 6-partite tournament H_1.

In this paper, we examine the cycle structure of a regular multipartite tournament and show that every arc in a regular n-partite tournament is in a cycle that contains vertices from exactly m partite sets for all m with $4 \leqslant m \leqslant n$. Moreover, if $3 \leqslant n \leqslant 5$ or if the cardinality common to all partite sets of a regular multipartite tournament is odd, then every arc is also in a cycle that contains vertices from exactly 3 partite sets. It is immediate that this result is a generalization of Alspach's theorem [1] for regular tournaments.

2. Main results

Theorem 1. *Let T be a regular n-partite tournament with $n \geqslant 3$. Then the following holds*:

(i) *Every arc of T is in a cycle which contains vertices from exactly* 3 *or exactly* 4 *partite sets.*

(ii) *If $n \leqslant 5$ or the cardinality common to the partite sets of T is odd*, *then every arc of T is in a cycle which contains vertices from exactly* 3 *partite sets.*

Proof. Let $V_1, V_2, \ldots, V_n$ be the partite sets of a regular n-partite tournament T. Note that all partite sets of T have the same number of vertices, say k. So, it is clear that

$$|N^+(x)| = |N^-(x)| = \frac{(n-1)k}{2} \quad \text{for every } x \in V(T).$$

Let ab be an arbitrary arc of T and assume without loss of generality that $a \in V_1$ and $b \in V_2$. Then, $V(T)$ can be partitioned into the following subsets:

$$A_1 = N^-(b) \cap V_1, \quad A_2 = N^+(b) \cap V_1,$$

$$B_1 = N^+(a) \cap V_2, \quad B_2 = N^-(a) \cap V_2,$$

$$X = N^-(a) \cap (V_3 \cup \cdots \cup V_n),$$

$$Y = N^+(a) \cap N^-(b) \cap (V_3 \cup \cdots \cup V_n),$$

$$Z = N^+(a) \cap N^+(b) \cap (V_3 \cup \cdots \cup V_n).$$

Suppose that ab is not in a cycle which contains vertices from exactly 3 partite sets. In particular, ab is not in a 3-cycle. Under this assumption, we first examine the domination relationships among the partition sets of $V(T)$ listed above.

From the regularity of T and the fact that ab is not in a 3-cycle, one can easily show that both X and Z are non-empty, and consequently, so are both A_2 and B_2. Furthermore, we have

$$X \to b, \quad \text{i.e., } N^-(a) \cap N^+(b) \cap (V_3 \cup \cdots \cup V_n) = \emptyset. \tag{1}$$

If there is an arc $a_2 \to x$ (resp. $z \to b_2$) from A_2 to X (resp. Z to B_2), then aba_2xa (resp. $abzb_2a$) is a cycle containing vertices from exactly 3 partite sets, a contradiction. Hence,

$$X \to A_2 \quad \text{and} \quad B_2 \to Z. \tag{2}$$

Let a_2 be a vertex in A_2. Since $|N^+(a_2)| = |N^+(a)|$ and $a \to b \to a_2$, there is a vertex u with $a_2 \to u \to a$. By (2), we see that $u \in B_2$. So,

$$N^+(v) \cap B_2 \neq \emptyset \quad \text{for every } v \in A_2. \tag{3}$$

Similarly, we can show that

$$N^-(u) \cap A_2 \neq \emptyset \quad \text{for every } u \in B_2. \tag{4}$$

From (3) and (4), we see that

$$X \to B_2 \quad \text{and} \quad A_2 \to Z. \tag{5}$$

To show $X \to A_1$, we assume that $|A_1| \geqslant 2$ and that $a_1 \to x$ for some $a_1 \in A_1 \setminus \{a\}$ and for some $x \in X$. Then, by (4), $a_1 \to B_2$ and consequently, $a_1 \to Z$ (otherwise, a cycle $abza_1b_2a$ yields a contradiction for some $z \in Z$ and for any $b_2 \in B_2$). Now, there must exist a vertex $b_1 \in B_1$ with $b_1 \to a_1$, which implies that $b_1 \to Z$. In fact, if such a b_1 does not exist, then $a_1 \to B_1$ and $|N^+(a_1)| \geqslant |Z| + |B_1| + |B_2| + |\{x\}| = |Z| + k + 1$, but $|N^+(b)| = |Z| + |A_2| \leqslant |Z| + k - 2$, which contradicts the regularity of T. From $|N^+(b)| = |N^+(b_1)|$, $b_1 \to Z$ and $b_1 \to a_1 \to b$, we see that A_2 contains a vertex a_2 with $b \to a_2 \to b_1$. But, a cycle $aba_2b_1a_1xa$ yields a contradiction. Therefore, $X \to A_1$.

If there exists a vertex $z \in Z$ which dominates some vertex a_1 in A_1, then it is easy to check that $B_2 \to a_1$. It follows that $|N^-(a_1)| \geqslant |X \cup B_2| + |\{z\}| = |N^-(a)| + 1$, a contradiction. So, we have $A_1 \to Z$.

By considering the converse T^{-1} of T, we have $Z \to B_1 \to X$ in T^{-1} from the argument above, that is, $X \to B_1 \to Z$ in T. So far, we have proved

$$X \to V_1 \cup V_2 \to Z, \tag{6}$$

under the assumption that ab is not in a cycle which contains vertices from exactly 3 partite sets. From the regularity of T, it is easy to see that

$$|Y| = |A_2| - |B_1| = |B_2| - |A_1| \leqslant (k-1) - 1 = k - 2. \tag{7}$$

It implies that $k \geqslant 2$.

To prove (i) we only need to show that there is at least one arc from Z to X. Suppose, on the contrary, that $N^+(z) \cap X = \emptyset$ for any $z \in Z$. Then, for any vertex $z \in Z$ (say, $z \in V_n$), the following holds:

$$\begin{aligned} |N^-(z)| &\geqslant |V_1 \cup V_2| + |N^-(z) \cap X| \\ &= 2k + |X \setminus V_n| \\ &\geqslant 2k + [|X| - (|V_n| - 1)] \end{aligned}$$

$$\geqslant 2k + \left[\frac{(n-3)k+2}{2} - (k-1)\right]$$

$$= \frac{(n-1)k}{2} + 2,$$

where the last inequality comes from $\frac{1}{2}(n-1)k = |N^-(a)| = |X| + |B_2| \leqslant |X| + (k-1)$. It gives a contradiction to the regularity of T. This proves (i).

Next, to prove (ii) we first let $n \leqslant 5$. Under the assumption that an arbitrarily given arc ab in T is not in a cycle which contains vertices from exactly 3 partite sets, the regularity of T and (6) imply that $n \geqslant 5$. Let $n = 5$. Then every vertex has exactly $2k$ out-neighbors and $2k$ in-neighbors. It follows from (6) again that there is no arc in the subdigraphs $T\langle X\rangle$ and $T\langle Z\rangle$. Thus, $|X|, |Z| \leqslant k$. By (7), T has at most $5k-2$ vertices, a contradiction.

Now, we assume that all partite sets of T have an odd number of vertices, i.e., k is odd. Then, it is clear that the subdigraph $T\langle V_1 \cup V_2\rangle$ is not regular. By (6), we conclude that $Y \neq \emptyset$. From (7) and the fact that $A_1, B_1 \neq \emptyset$, we see that $|A_2|, |B_2| \geqslant 2$.

Let y be an arbitrary vertex of Y. Suppose that A_2 contains two vertices a_2 and a_2' with $a_2 \to y \to a_2'$. By (3), we have $a_2' \to b'$ for some $b' \in B_2$, and hence, a cycle $aba_2ya_2'b'a$ yields a contradiction. This means that either $A_2 \to y$ or $y \to A_2$. Similarly, one can verify by (4) that either $B_2 \to y$ or $y \to B_2$ holds.

If $A_2 \to y$, then it is easy to see that $B_2 \to y$. Assume now that $y \to A_2$ and $B_2 \to y$. From (3) and (4) and $|A_2|, |B_2| \geqslant 2$, it is not difficult to confirm that there are two vertex-disjoint arcs from A_2 to B_2, say $a_2 \to b_2$ and $a_2' \to b_2'$. But now, a cycle $aba_2b_2ya_2'b_2'a$ yields a contradiction. This implies that if $y \to A_2$, then $y \to B_2$ as well. So, we have proved that for each vertex $y \in Y$, either $A_2 \cup B_2 \to y$ or $y \to A_2 \cup B_2$. Thus, Y can be partitioned into two subsets Y_1 and Y_2 with $Y_2 \to A_2 \cup B_2 \to Y_1$.

If there is an arc $y_1 \to a_1$ from Y_1 to A_1, then, from the assumption that ab is not in a cycle containing vertices from exactly 3 partite sets of T, we see that $B_2 \to a_1$. But now, $|N^-(a_1)| > |N^-(a)|$ holds, a contradiction. Therefore, we have $A_1 \to Y_1$.

To show $B_1 \setminus \{b\} \to Y_1$, we assume on the contrary that there is an arc $y_1 \to b_1$ from Y_1 to $B_1 \setminus \{b\}$. Because of $|N^-(b_1)| = |N^-(a)|$, there is a vertex $u \in V_1 \setminus \{a\}$ with $b_1 \to u$. From $|A_2| \geqslant 2$ and (3), it is easy to check that $u \in A_1 \setminus \{a\}$. By $|N^-(u)| = |N^-(a)|$, we conclude that u has at least one out-neighbor, say b_2, in B_2. Now we see that a cycle $aba_2y_1b_1ub_2a$ for any $a_2 \in A_2$ yields a contradiction.

In summary, we have

$$V_1 \cup (V_2 \setminus \{b\}) \to Y_1. \tag{8}$$

By considering the converse T^{-1} of T, we also have

$$Y_2 \to (V_1 \setminus \{a\}) \cup V_2. \tag{9}$$

Let $T' = T\langle V_1 \cup V_2\rangle$ and let $d^+_{T'}(x) = |N^+(x) \cap V(T')|$ for $x \in V(T')$. From (6), (8) and (9) and the regularity of T, we have $d^+_{T'}(x) = |N^+(a) \cap (V_1 \cup V_2 \cup Y_2)| = |B_1| + |Y_2|$ for every $x \in V(T') \setminus \{a,b\}$. Moreover, the first part of (7) implies $d^+_{T'}(b) = |A_2| = |B_1| + |Y| = |B_1| + |Y_1| + |Y_2|$. Since

$$\sum_{x \in V(T')\setminus\{a,b\}} d^+_{T'}(x) + d^+_{T'}(a) + d^+_{T'}(b) = |E(T')|,$$

we have $(2k-2)(|B_1| + |Y_2|) + |B_1| + (|B_1| + |Y_1| + |Y_2|) = k^2$. It follows that

$$|B_1| = \frac{k}{2} - |Y_2| + \frac{|Y_2| - |Y_1|}{2k}.$$

By (7), we see that

$$-\frac{1}{2} < \frac{-|Y_1|}{2k} \leqslant \frac{|Y_2| - |Y_1|}{2k} \leqslant \frac{|Y_2|}{2k} < \frac{1}{2}.$$

Since k is odd, the second last equation thus has no integer solution for $|B_1|$. This impossibility completes the proof of (ii). □

At the beginning of the proof of Theorem 1 (up to (3)), we actually showed the following lemma, which will be used for the next theorem.

Lemma 2. *Any arc ab of a regular n-partite tournament with $n \geqslant 3$ is in at least one of the following cycles*:
(i) *A* 3-*cycle*.
(ii) *A* 4-*cycle of the type $aba'ua$, where a, a' are in the same partite set and b, u are in different partite sets*.
(iii) *A* 4-*cycle of the type $aba'b'a$, where a and a' (resp. b and b') are in the same partite set*.

Theorem 3. *Let T be a regular n-partite tournament with $n \geqslant 3$. If an arc of T is in a cycle that contains vertices from exactly m partite sets for some m with $3 \leqslant m < n$, then it is also in a cycle that contains vertices from exactly $m+1$ partite sets.*

Proof. Let $V_1, V_2, \ldots, V_n$ be the partite sets of T and let v_1v_2 be an arc that is in a cycle, say $C = v_1v_2 \cdots v_tv_1$, which contains vertices from exactly m partite sets for some $3 \leqslant m < n$. Assume without loss of generality that $v_i \in V_i$ for $i = 1, 2$ and $V(C) \cap V_j = \emptyset$ for $j > m$. Let

$$X = N^-(v_1) \cap (V_{m+1} \cup \cdots \cup V_n) \quad \text{and} \quad Y = N^+(v_1) \cap (V_{m+1} \cup \cdots \cup V_n).$$

It is clear that $X \cup Y = V_{m+1} \cup \cdots \cup V_n$ and every vertex of $X \cup Y$ is adjacent with all vertices in C. We consider the following two cases.

Case 1: $X \neq \emptyset$.

If there is a vertex x in X such that $v_t \to x$, then a cycle $v_1v_2 \cdots v_txv_1$ is a desired cycle, that is, it is a cycle that contains vertices from exactly $m+1$ partite sets.

If such vertex x does not exist, then $X \to v_t$. Since $X \to \{v_1, v_t\}$, if some $v_i \in V(C)$ dominates a vertex x in X, then we can construct a cycle having vertices $v_1, v_2, \ldots, v_t$ and x, which contains the arc v_1v_2 and vertices from exactly $m+1$ partite sets. Now, we assume that $X \to V(C)$.

Let t' be an integer such that the path $v_1v_2 \cdots v_{t'}$ contains vertices from exactly $m-1$ partite sets. Clearly, $2 \leqslant t' < t$. Let x be a fixed vertex of X and assume, without loss of generality, that $x \in V_n$. According to Lemma 2, we need to consider only the following three cases for the arc $xv_{t'}$.

The first case is that $xv_{t'}$ is in a 3-cycle, say $xv_{t'}yx$. If $y \in V_i$ for some $i \leqslant m$, then $v_1 \cdots v_{t'}yxv_{t'+1} \cdots v_tv_1$ is a desired cycle. If $y \in V_i$ for some $i > m$, then $y \in Y$ and $v_1v_2 \cdots v_{t'}yxv_1$ is a cycle that contains vertices from exactly $m+1$ partite sets.

The second case is that $xv_{t'}$ is in a cycle $xv_{t'}x'ux$, where x' is in V_n and $v_{t'}, u$ are in different partite sets. Clearly, x' and u do not belong to C. If $u \in X \cup Y$, then $v_1v_2 \cdots v_{t'}x'uxv_1$ is a desired cycle. Otherwise, u belongs to V_ℓ for some $\ell \leqslant m$, and $v_1 \cdots v_{t'}x'uxv_{t'+1} \cdots v_tv_1$ is a desired cycle.

The third case is that $xv_{t'}$ is in a cycle $xv_{t'}x'vx$, where $x' \in V_n$ and $v_{t'}, v$ are in the same partite set. Since x' and v cannot be in C, we see that $v_1v_2 \cdots v_{t'}x'vxv_{t'+1} \cdots v_tv_1$ is a desired cycle.

Case 2: $X = \emptyset$.

Note that $v_1 \to Y$ and $Y = V_{m+1} \cup \cdots \cup V_n$ in this case.

Suppose that Y contains a vertex y (assume without loss of generality that $y \in V_n$) with $v_i \to y$ for $i=2$ or 3. If $y \to v_j$ for some $j > i$, then we see as in Case 1 that y can be inserted into C to form a desired cycle. So, we may assume that $v_j \to y$ for all $j > i$. Since $v_1 \to V_n$ and $|N^-(y)| = |N^-(v_i)|$ by the regularity, v_1y is in a 3-cycle, say v_1yzv_1. Clearly, $z \notin Y \cup V(C)$, and hence, we find a desired cycle $v_1v_2 \cdots v_tyzv_1$ again.

Suppose now that $Y \to \{v_2, v_3\}$. If Y contains a vertex y' (assume without loss of generality that $y' \in V_n$) with $y' \to v_t$, then either y' can be inserted into C to form a desired cycle or $y' \to v_i$ for all $i \geqslant 2$. Now we consider the last case. From the fact that $V_n \to v_2$ and $|N^+(y')| = |N^+(v_2)|$, we see that $y'v_2$ is in a 3-cycle, say $y'v_2wy'$. Clearly, $w \notin V(C) \cup Y$, and hence $v_1v_2wy'v_3 \cdots v_tv_1$ is a desired cycle. Therefore, we may assume that $v_t \to Y$. Note that $t \geqslant 4$ now.

Let $W = N^+(v_2) - (V(C) \cup V_1)$. It is not difficult to check that if there is an arc from W to Y, or if $Y \to W$ and there is an arc from W to v_1, one can find a desired cycle. So, we consider the case when $Y \cup \{v_1\} \to W$.

If there is a vertex v_i in C with $v_2 \to v_i$ and $v_{i-1} \to v_1$, then $i \geqslant 4$ and for any $y \in Y$, $v_1v_2v_i \cdots v_tyv_3 \cdots v_{i-1}v_1$ is a desired cycle. Thus, we assume that if $v_2 \to v_j$, then either $v_1 \to v_{j-1}$ or $v_{j-1} \in V_1$. This implies that

$$|N^+(v_1) \cap V(C)| \geqslant |N^+(v_2) \cap V(C)| - |(V_1 \setminus \{v_1\}) \cap V(C)|.$$

Note that this inequality implies the following:

$$\begin{aligned}|N^+(v_1)| &\geqslant |Y| + |W| + |N^+(v_1) \cap V(C)| \\ &\geqslant |Y| + |W| + |N^+(v_2) \cap V(C)| - |(V_1 \setminus \{v_1\}) \cap V(C)|\end{aligned}$$

$$\geqslant |V_n| + |N^+(v_2)| - |V_1 \setminus \{v_1\}|$$
$$= |N^+(v_2)| + 1.$$

This contradiction to the regularity of T completes the proof. □

Combining Theorems 1 and 3, we have the following corollaries.

Corollary 4. *Let T be a regular n-partite tournament with $3 \leqslant n \leqslant 5$. Then every arc of T is in a cycle that contains vertices from exactly m partite sets for all m with $3 \leqslant m \leqslant n$.*

Corollary 5. *Let T be a regular n-partite tournament with $n \geqslant 6$. Then every arc of T is in a cycle that contains vertices from exactly m partite sets for all m with $4 \leqslant m \leqslant n$.*

Corollary 6. *Let T be a regular n-partite tournament with $n \geqslant 3$. If the cardinality common to all partite sets of T is odd, then every arc of T is in a cycle that contains vertices from exactly m partite sets for all m with $3 \leqslant m \leqslant n$.*

Finally, Alspach's theorem follows from Corollary 6.

Corollary 7 (Alspach [1]). *A regular tournament is arc-pancyclic.*

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